# THE SECOND APPROXIMATION IN THE ASYMPTOTIC THEORY OF SONIC FLOW OF A REAL GAS PAST BODIES OF REVOLUTION

PMM Vol. 31, No. 5, 1967, pp. 783-793
V. N. DIESPEROV and O. S. RYZHOV
(Moscow)
(Received May 22, 1967)

The asymptotic laws of damping of perturbations induced in a sonic stream of dissipating gas by bodies of revolution were derived in paper [1]. These laws were shown to be different from those obtaining when the medium viscosity and thermal conductivity coefficients are assumed to be zero [2 to 5]. The conclusions arrived at in [1] were based on series expansions of the unknown functions with respect to two small independent parameters, which characterize the longitudinal and the transverse components of the perturbation velocity. The exact relationship between these two small parameters could not be established with the aid of the first approximation theory which yields an estimate only of this relationship. This difficulty is eliminated in the second approximation theory.

**1. Analysis of equations.** Let $x$ and $r$ denote the axes of a cylindrical coordinate system, $v_x$ and $v_r$ the velocity vector components along these axes, $\rho$ the density, $p$ the pressure, $s$ the specific entropy, $T$ the temperature, $\lambda_1$ the viscosity coefficient, $\lambda_2$ the secondary viscosity coefficient, and $k$ the thermal conductivity coefficient. We shall write the Navier-Stokes and the heat transfer continuity equations on the assumption of the gas flow symmetry relative to the $x$-axis in the form [6]

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_r}{\partial r} + \frac{\rho v_r}{r} = 0 \qquad (1.1)$$

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\lambda_1 \frac{\partial v_x}{\partial x} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right] +$$
$$+ \frac{\partial}{\partial r}\left[\lambda_1\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{\lambda_1}{r}\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) \qquad (1.2)$$

$$\rho\left(v_x \frac{\partial v_r}{\partial x} + v_r \frac{\partial v_r}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial x}\left[\lambda_1\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] +$$
$$+ \frac{\partial}{\partial r}\left[2\lambda_1 \frac{\partial v_r}{\partial r} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right] + \frac{2\lambda_1}{r}\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r}\right) \qquad (1.3)$$

$$\rho T\left(v_x \frac{\partial s}{\partial x} + v_r \frac{\partial s}{\partial r}\right) = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial r}\left(k \frac{\partial T}{\partial r}\right) + \frac{k}{r}\frac{\partial T}{\partial r} + 2\lambda_1\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)^2 + \left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2\right] + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)^2 \qquad (1.4)$$

Entropy and temperature may be eliminated from this system by using the following thermodynamic relationships [7]

$$ds = \frac{c_p}{\alpha\rho a^2 T}(dp - a^2 d\rho), \qquad dT = \frac{1}{\alpha\rho a^2}(\varkappa dp - a^2 d\rho)$$

$$\left(\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p, \ a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s, \ V = \frac{1}{\rho}, \ \varkappa = \frac{c_p}{c_v}\right) \qquad (1.5)$$

Here, $V = 1/\rho$ denotes the specific volume, $\alpha$ the thermal expansion coefficient, $a$ the adiabatic velocity of sound, $c_p$ the specific heat at constant pressure and $c_v$ the specific heat at constant volume. When solving problems of gas dynamics, the medium viscosity and thermal conductivity coefficients are often taken to be dependent on temperature only. This assumption will be used in the following text. Pressure and density are selected as the independent thermodynamic variables, the remaining ones being functions of these two in accordance with Formulas (1.5).

We shall assume that in the area of space under consideration the values of all gas parameters differ only insignificantly from those in the free steady-state stream. We also assume that the gas particles velocity coincides in magnitude with the velocity of sound, and is directed along the $x$-axis. Parameters of the unperturbed medium will be denoted by an asterisk, and the characteristic length along the $x$-axis by $L$. Using the results of [1] we introduce the following independent dimensionless variables and expansions of the unknown functions:

$$x = Lx', \qquad r = (L / \Delta)\, r' \qquad (1.6)$$

$$v_x = a_* \left[1 + \varepsilon\,(v_{x1} + \delta v_{x2} + \ldots)\right], \qquad v_r = \varepsilon\Delta a_* \,(v_{r1} + \delta v_{r2} + \ldots)$$

$$\rho = \rho_* \left[1 + \varepsilon\,(\rho_1 + \delta\rho_2 + \ldots)\right], \qquad p = p_* \left[1 + \varepsilon\,(p_1 + \delta p_2 + \ldots)\right]$$

Here, $\varepsilon$, $\delta$ and $\Delta$ are numerical parameters considerably smaller than unity. The substitution of relationships (1.6) into the system of Eqs.(1.1) to (1.4) yields three dimensionless coefficients, viz. Reynolds and Péclet numbers

$$N_{\mathrm{Re1}} = \frac{\rho_* a_* L}{\lambda_{1*}}, \qquad N_{\mathrm{Re2}} = \frac{\rho_* a_* L}{\lambda_{2*}}, \qquad N_{\mathrm{Pe}} = \frac{\rho_* a_* c_{p*} L}{k_*}$$

These numbers are computed from values of the gas-dynamical functions of the free sonic stream. We assume the reciprocal values of these numbers to be of the same order of magnitude, and considerably smaller than unity. In the derivation of approximate equations we shall retain in all relationships terms of the first and second order only, neglecting those of a higher order of smallness.

Substituting Formulas (1.6) into the continuity equation we obtain (*)

$$\frac{\partial}{\partial x}(\rho_1 + v_{x1}) + \varepsilon\frac{\partial \rho_1 v_{x1}}{\partial x} + \delta\frac{\partial}{\partial x}(\rho_2 + v_{x2}) + \Delta^2\left(\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r}\right) = 0 \qquad (1.7)$$

The projection of the Navier-Stokes Eq.(1.2) on to the $x$-axis yields

$$\frac{\partial}{\partial x}\left(v_{x1} + \frac{p_*}{\rho_* a_*^2} p_1\right) + \varepsilon(\rho_1 + v_{x1})\frac{\partial v_{x1}}{\partial x} +$$
$$+ \delta\frac{\partial}{\partial x}\left(v_{x2} + \frac{p_*}{\rho_* a_*^2} p_2\right) - \frac{1}{N_{\mathrm{Re}}}\frac{\partial^2 v_{x1}}{\partial x^2} = 0 \qquad (1.8)$$

*) The total Reynolds number $N_{\mathrm{Re}}$ appearing in this equation is associated with the

---

*) Primes over independent dimensionless variables are omitted here and in the following text.

so-called "longitudinal viscosity"

$$\frac{1}{N_{\mathrm{Re}}}=\frac{4}{3}\frac{1}{N_{\mathrm{Re1}}}+\frac{1}{N_{\mathrm{Re2}}}$$

and the reciprocal of this number is, by virtue of the above assumptions considerably smaller than unity.

Projecting the Navier-Stokes Eqs. (1.3) on to the $r$-axis we derive the following relationship:

$$\frac{\partial v_{r1}}{\partial x}+\frac{p_*}{\rho_* a_*^2}\frac{\partial p_1}{\partial r}+\varepsilon(\rho_1+v_{x1})\frac{\partial v_{r1}}{\partial x}+\delta\left(\frac{\partial v_{r2}}{\partial x}+\frac{p_*}{\rho_* a_*^2}\frac{\partial p_2}{\partial r}\right)-$$

$$-\frac{1}{N_{\mathrm{Re}}}\frac{\partial^2 v_{x1}}{\partial x\,\partial r}+\frac{1}{N_{\mathrm{Re}}}\frac{\partial}{\partial x}\left(\frac{\partial v_{x1}}{\partial r}-\frac{\partial v_{r1}}{\partial x}\right)=0 \qquad (1.9)$$

Equating the principal terms of Eqs. (1.7) and (1.8) to zero, and integrating the expressions thus obtained, we have

$$\rho_1=\frac{p_*}{\rho_* a_*^2}\,p_1=-v_{x1} \qquad (1.10)$$

These formulas make it possible to simplify equations of the second approximation

$$\delta\frac{\partial}{\partial x}(\rho_2+v_{x2})=2\varepsilon v_{x1}\frac{\partial v_{x1}}{\partial x}-\Delta^2\left(\frac{\partial v_{r1}}{\partial r}+\frac{v_{r1}}{r}\right) \qquad (1.11)$$

$$\delta\left(v_{x2}+\frac{p_*}{\rho_* a_*^2}\,p_2\right)=\frac{1}{N_{\mathrm{Re}}}\frac{\partial v_{x1}}{\partial x} \qquad (1.12)$$

Let us consider Eq. (1.9). A substitution into it of the functional relationship defined by Expressions (1.10) and (1.12) shows that the flows under consideration are irrotational not only in the first [1], but also in the second approximation.

$$\frac{\partial v_{x1}}{\partial r}=\frac{\partial v_{r1}}{\partial x},\qquad \frac{\partial v_{x2}}{\partial r}=\frac{\partial v_{r2}}{\partial x} \qquad (1.13)$$

When considering the heat transfer equation it is necessary to take into account the effects of dissipative factors directly in the first approximation. A preliminary transformation of this equation is required in order to eliminate quantities of the first order of smallness related to mass and impulse transfer. We denote the right-hand side of Eq. (1,4) by $L(k, \lambda_1, \lambda_2)$, and by $L_x(\lambda_1, \lambda_2)$ and $L_r(\lambda_1, \lambda_2)$ the right-hand sides of Eqs. (1.2) and (1.3), respectively, omitting their first terms. The required relationship may be conveniently expressed by

$$\rho\left[(v_x^2-a^2)\frac{\partial v_x}{\partial x}+v_x v_r\left(\frac{\partial v_x}{\partial r}+\frac{\partial v_r}{\partial x}\right)+(v_r^2-a^2)\frac{\partial v_r}{\partial r}-\frac{a^2 v_r}{r}\right]=$$

$$=v_x L_x(\lambda_1,\lambda_2)+v_r L_r(\lambda_1,\lambda_2)-\frac{\alpha a^2}{c_p}L(k,\lambda_1,\lambda_2) \qquad (1.14)$$

The velocity of sound $a$ appears in the coefficients of this equation. Its expansion into a Taylor series is most easily carried out, if pressure and entropy are taken as independent variables, because the variation of the latter in a perturbed flow field is of a higher order of smallness than that of variation of all of the remaining thermodynamic functions. Taking note of this we write

$$a=a_*+\left(\frac{\partial a}{\partial p_*}\right)_s(p-p_*)+\frac{1}{2}\left(\frac{\partial^2 a}{\partial p_*^2}\right)_s(p-p_*)^2+\left(\frac{\partial a}{\partial s_*}\right)_p(s-s_*)+\cdots \qquad (1.15)$$

The variation of entropy is readily determined from Eq. (1.4) in which the principal terms only are to be retained. In terms of initial physical variables

$$\frac{\partial s}{\partial x} = \frac{1}{N_{\mathrm{Pe}}} \frac{c_{p*} L}{T_*} \frac{\partial^2 T}{\partial x^2}$$

Integration of the latter equality, with Formulas (1.5) and (1.10) taken into account yields

$$s = s_* \left[ 1 - \frac{\varepsilon}{N_{\mathrm{Pe}}} \frac{c_{p*}(\varkappa_* - 1)}{\alpha_* s_* T_*} \frac{\partial v_{x1}}{\partial x} \right]$$

We introduce the following dimensionless coefficients

$$m_1 = \frac{1}{2\rho^3 a^2} \left( \frac{\partial^2 p}{\partial V^2} \right)_s, \qquad m_2 = \frac{1}{2\rho^4 a^2} \left( \frac{\partial^3 p}{\partial V^3} \right)_s \tag{1.16}$$

which define the adiabatic compression of the gas, and coefficient

$$m_3 = \frac{c_p (\varkappa - 1)}{\alpha a T} \left( \frac{\partial a}{\partial s} \right)_p \tag{1.17}$$

which shows the rate of increase of the velocity of sound with increasing entropy at constant pressure. Let the Prandtl number

$$N_{\mathrm{Pr}} = \frac{N_{\mathrm{Pe}}}{N_{\mathrm{Re}}}$$

denote the ratio of the Péclet and Reynolds numbers which in accordance with the assumptions made above is of the order of unity. With these notations the expansion (1.15) of the velocity of sound may be expressed thus

$$a = a_* [1 - \varepsilon (m_{1*} - 1) v_{x1} - {}^1/_2 \varepsilon^2 (3 m_{1*}{}^2 + m_{2*}) v_{x1}{}^2 - \varepsilon\delta (m_{1*} - 1) v_{x2} + $$
$$+ \frac{\varepsilon}{N_{\mathrm{Re}}} \left( m_{1*} - 1 - \frac{m_{3*}}{N_{\mathrm{Pr}}} \right) \frac{\partial v_{x1}}{\partial x} \Big]$$

We shall simplify now Eq. (1.14). In addition to coefficients (1.16) and (1.17) we shall use the following dimensionless thermodynamic parameters:

$$m_4 = \frac{\alpha a^2}{c_p}, \qquad m_5 = \frac{c_p (\varkappa - 1)^2}{\alpha^2 a^2} \left( \frac{\partial}{\partial T} \frac{\alpha a^2}{c_p} \right)_s$$

and

$$\mu_1 = \frac{\varkappa - 1}{\alpha} \frac{{}^4/_3 d\lambda_1 / dT + d\lambda_2 / dT}{{}^4/_3 \lambda_1 + \lambda_2}$$

$$\mu_2 = \frac{\rho (\varkappa - 1)}{k} \left( \frac{\partial}{\partial T} \frac{k}{a\rho} \right)_s, \qquad \mu_3 = \frac{\rho a^2 (\varkappa - 1)}{k} \left( \frac{\partial}{\partial T} \frac{\varkappa k}{\alpha \rho a^2} \right)_s$$

which characterize the dependence of the medium viscosity and thermal conductivity coefficients on temperature. We shall assume the order of magnitude of all coefficients $m_1$ to $m_5$, and $\mu_1$ to $\mu_3$ to be equal to unity. We shall present the results of transformation of Eq. (1.14) directly in its final form, going over from the gas density and pressure to projections on the $x$- and $r$-axes of the perturbed velocity vector using relationships (1.10) and (1.13). This, of course, is not the most general form of the sought equation, it is, nevertheless, adequate for the objectives set out, and is comparatively simple.

Thus, we have

$$2\varepsilon m_{1*} v_{x1} \frac{\partial v_{x1}}{\partial x} - \Delta^2 \left( \frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} \right) - \frac{1}{N_{\mathrm{Re}}} \left( 1 + \frac{\varkappa_* - 1}{N_{\mathrm{Pr}}} \right) \frac{\partial^2 v_{x1}}{\partial x^2} +$$

$$+\varepsilon^2(2m_{1*}^2+m_{2*})v_{x1}^2\frac{\partial v_{x1}}{\partial x}+2\varepsilon\delta m_{1*}\left(v_{x1}\frac{\partial v_{x2}}{\partial x}+v_{x2}\frac{\partial v_{x1}}{\partial x}\right)+$$

$$+\varepsilon\Delta^2\left[2v_{r1}\frac{\partial v_{x1}}{\partial r}+(2m_{1*}-1)v_{x1}\left(\frac{\partial v_{r1}}{\partial r}+\frac{v_{r1}}{r}\right)\right]-\delta\Delta^2\left(\frac{\partial v_{r2}}{\partial r}+\frac{v_{r2}}{r}\right)-$$

$$-\frac{\varepsilon}{N_{\rm Re}}\left[\left(1-\mu_{1*}+\frac{\nu_{1*}}{N_{\rm Pr}}\right)v_{x1}\frac{\partial^2 v_{x1}}{\partial x^2}-\left(\nu_{2*}-\frac{\nu_{3*}}{N_{\rm Pr}}\right)\left(\frac{\partial v_{x1}}{\partial x}\right)^2\right]-$$

$$-\frac{\delta}{N_{\rm Re}}\left(1+\frac{\varkappa_*-1}{N_{\rm Pr}}\right)\frac{\partial^2 v_{x2}}{\partial x^2}-\frac{\Delta^2}{N_{\rm Re}}\left(1+\frac{\varkappa_*-2}{N_{\rm Pr}}\right)\frac{\partial}{\partial x}\left(\frac{\partial v_{r1}}{\partial r}+\frac{v_{r1}}{r}\right)+$$

$$+\frac{1}{N_{\rm Re}^2}\frac{\varkappa_*}{N_{\rm Pr}}\frac{\partial^3 v_{x1}}{\partial x^3}=0 \qquad (1.18)$$

$$(\nu_1=2+\mu_2-\mu_3-m_5,\ \nu_2=2+\mu_1+m_4-2m_1,\ \nu_3=2+\mu_2-\mu_3-2m_3)$$

A further transformation of Eq. (1.18) may be achieved by various means, the seclection of which is dependent on the relative magnitude of the small parameters $\epsilon$, $\delta$, $\Delta$, and $N_{\rm Re}^{-1}$ appearing in it. An analysis of equations of the first approximation was given in paper [1]. On the basis of results of that paper we shall construct in the second approximation a pattern of flow past a body of revolution placed in a sonic stream of a dissipative gas.

**2. The asymptotic laws of perturbation damping.** When the velocity field at considerable distances from the body of finite dimensions is considered, it can be assumed that $\varepsilon \ll \Delta^2 \sim N_{\rm Re}^{-1}$. Assuming for simplicity's sake that

$$\Delta^2=\frac{1}{N_{\rm Re}}\left(1+\frac{\varkappa_*-1}{N_{\rm Pr}}\right)$$

we derive from (1.18) the missing equation of the first approximation [1]

$$\frac{\partial^2 v_{x1}}{\partial x^2}+\frac{\partial v_{r1}}{\partial r}+\frac{v_{r1}}{r}=0 \qquad (2.1)$$

which together with Eq. (1.13) constitutes a closed system. Its solution, which defines the asymptotic laws of perturbation damping at a distance from a body of circular cross section, is of the form

$$v_{x1}=r^{-n}f_1(\xi),\qquad v_{r1}=r^{-n-1/3}g_1(\xi),\qquad \xi=xr^{-2/3} \qquad (2.2)$$

In Formulas (2.2) the exponent $n = 4/3$. If we denote the Euler gamma-function by $\Gamma(\alpha)$, and the confluent hypergeometric function by $\Phi(\beta,\gamma,\eta)$, then [1]

$$f_1=c_1\left[\Phi(^2/_3,{}^1/_3;\eta)-\frac{\Gamma^2(^1/_3)}{2\Gamma^2(^2/_3)}\eta^{2/3}\Phi(^4/_3,{}^5/_3;\eta)\right] \qquad (2.3)$$

$$g_1=2^{4/3}c_1\left[\eta^{1/3}\Phi(^5/_3,{}^4/_3;\eta)-\frac{\Gamma^2(^1/_3)}{6\Gamma^2(^2/_3)}\Phi(^4/_3,{}^2/_3;\eta)\right] \qquad (2.4)$$

$$\eta=-(^4/_{27})\xi^3$$

The integral of (2.2) to (2.4) corresponds to a point source at the coordinate origin, with a sonic stream uniform at infinity flowing past it. Constant $c_1$ is proportional to the source power $Q$. As previously stated, the theory of the first approximation allows to obtain only the estimate $\epsilon/\Delta \ll \Delta$ of the ratio of the two small parameters $\epsilon$ and $\Delta$. In order to obtain the exact relationship between these two parameters, it is necessary to consider the velocity field pattern in the second approximation.

We simplify Eq. (1.18) with the aid of Formulas (2.1) and (2.2), retaining terms of one order of smallness only

$$\delta\Delta^2\left(\frac{\partial^2 v_{x2}}{\partial x^2}+\frac{\partial v_{r2}}{\partial r}+\frac{v_{r2}}{r}\right)=2\varepsilon m_{1*}v_{x1}\frac{\partial v_{x1}}{\partial x}-$$

$$-\frac{\Delta^2}{N_{\mathrm{Re}}}\left(1+\frac{\varkappa_*-2}{N_{\mathrm{Pr}}}\right)\frac{\partial}{\partial x}\left[\frac{\partial v_{r1}}{\partial r}+\frac{v_{r1}}{r}\right]+\frac{1}{N_{\mathrm{Re}}^2}\frac{\varkappa_*}{N_{\mathrm{Pr}}}\frac{\partial^3 v_{x1}}{\partial x^3} \qquad (2.5)$$

All coefficients of the derived equation must be of the order of $\epsilon$. Hence, we conclude $\Delta\sim\varepsilon^{1/4}$, $\delta\sim\varepsilon^{1/2}$ and $N_{\mathrm{Re}}^{-1}\sim\varepsilon^{1/2}$. These relationships are in full accord with previous results. Numerical values of coefficients in Eq. (2.5) are the simplest when

$$\Delta=\varepsilon^{1/4}(2m_{1*})^{1/4}A^{-1/4}\left(1+\frac{\varkappa_*-1}{N_{\mathrm{Pr}}}\right)^{1/2},\qquad \delta=\varepsilon^{1/2}(2m_{1*})^{1/2}A^{1/2}\left(1+\frac{\varkappa_*-1}{N_{\mathrm{Pr}}}\right)^{-1}$$

$$\frac{1}{N_{\mathrm{Re}}}=\varepsilon^{1/2}(2m_{1*})^{1/2}A^{-1/2},\qquad A=\frac{\varkappa_*}{N_{\mathrm{Pr}}}+\left(1+\frac{\varkappa_*-2}{N_{\mathrm{Pr}}}\right)\left(1+\frac{\varkappa_*-1}{N_{\mathrm{Pr}}}\right)$$

Substituting in Eq. (2.5) the second derivative $\partial^2 v_{x1}/\partial x^2$ of function $v_{x1}(x,r)$ for the expression in brackets, we find

$$\frac{\partial^2 v_{x2}}{\partial x^2}+\frac{\partial v_{r2}}{\partial r}+\frac{v_{r2}}{r}=v_{x1}\frac{\partial v_{x1}}{\partial x}+\frac{\partial^3 v_{x1}}{\partial x^3} \qquad (2.6)$$

Eq. (2.6) together with Eq. (1.13) constitutes a closed system which is used in the determination of projections on the $x$- and $r$-axes of components of the perturbed velocity vector. We note that the homogeneous equations corresponding to this system coincide exactly with the first Eqs. (1.13) and with Eq. (2.1) for functions $v_{x1}(x,r)$ and $v_{r1}(x,r)$ of the first approximation.

This feature makes the integration of the second of Eqs. (1.13) and of Eq. (2.6) considerably easier.

Having obtained the solution of these, we find functions $\rho_2(x,r)$ and $p_2(x,r)$ from the following relationships:

$$\rho_2=\frac{1}{A}\left(1+\frac{\varkappa_*-1}{N_{\mathrm{Pr}}}\right)^2\frac{\partial v_{x1}}{\partial x}-v_{x2}$$

$$p_2=\frac{\rho_* a_*^2}{p_*}\left[\frac{1}{A}\left(1+\frac{\varkappa_*-1}{N_{\mathrm{Pr}}}\right)\frac{\partial v_{x1}}{\partial x}-v_{x2}\right]$$

which follow from (1.11) and (1.12). We shall now proceed with the integration of the second of Eqs. (1.13) and of Eq. (2.6). The combination appearing in the right-hand side of Eq. (2.6)

$$v_{x1}\frac{\partial v_{x1}}{\partial x}+\frac{\partial^3 v_{x1}}{\partial x^3}=\frac{1}{r^{10/3}}\left(f_1\frac{df_1}{d\xi}+\frac{d^3 f_1}{d\xi^3}\right)$$

where function $f_1(\xi)$ is specified by relationships (2.3) and (2.4). It follows from this that magnitudes $v_{x2}(x,r)$ and $v_{r2}(x,r)$ may, as previously, be sought in the form of (2.2) with $n=2$. Functions $f_2(\xi)$ and $g_2(\xi)$ satisfy the ordinary differential equations

$$\frac{d^2 f_2}{d\xi^2}-\frac{2}{3}\xi\frac{dg_2}{d\xi}-\frac{4}{3}g_2=f_1\frac{df_1}{d\xi}+\frac{d^3 f_1}{d\xi^3},\qquad \frac{2}{3}\xi\frac{df_2}{d\xi}+\frac{dg_2}{d\xi}+2f_2=0 \quad (2.7)$$

The integrating factor of the first of Eqs. (2.7) is $\xi$. Noting this we derive, after some simple transformations

$$\xi \frac{df_2}{d\xi} - f_2 - \frac{2}{3}\xi^2 g_2 = \omega(\xi), \qquad \frac{dg_2}{d\xi} + \frac{8}{3} f_2 + \frac{4}{9}\xi^2 g_2 = -\frac{2}{3}\omega(\xi)$$

$$\omega(\xi) = \xi \frac{d^2 f_1}{d\xi^2} - \frac{df_1}{d\xi} + \frac{1}{2}\left[\xi f_1^2(\xi) - \int_{-\infty}^{\xi} f_1^2(\xi)\, d\xi\right] \qquad (2.8)$$

In order to satisfy the natural symmetry condition of the stream $v_r(x,r) \to 0$, when $r \to 0$, and $x < 0$, the constant of integration is selected equal to zero. The area downstream of the body is occupied by a vortex trail, therefore the two components $v_x(x,r)$ and $v_r(x,r)$ of the perturbed velocity vector may, generally speaking, have singularities when $r \to 0$, and $x > 0$, in spite of the absence of such in the first approximation theory. We shall denote by $f_2^0(\xi)$ and $g_2^0(\xi)$ functions which satisfy the homogeneous equations corresponding to system (2.8). Components of the perturbed velocity vector derived with the aid of these functions must vanish at infinity, except in the area of the vortex trail. The solution of the homogeneous system of Eqs. (2.8) which satisfies the latter condition may be easily written now with the use of results of paper [1], and when $r \to 0$ it is in fact a regular one for any values of $x \neq 0$.

With an accuracy of the order of the arbitrary factor we have

$$f_2^0 = \frac{df_1}{d\xi}, \qquad g_2^0 = \frac{dg_1}{d\xi}$$

Hence, in accordance with rules of confluent hypergeometric function integration, we have

$$f_2^0 = c_2 \eta^{1/3}\left[\Phi(4/3, 2/3; \eta) - \frac{6\Gamma^2(2/3)}{\Gamma^2(1/3)}\eta^{1/3}\Phi(5/3, 4/3; \eta)\right]$$

$$g_2^0 = 2^{4/3} c_2 \left[\eta^{2/3}\Phi(7/3, 5/3; \eta) - \frac{\Gamma^2(2/3)}{\Gamma^2(1/3)}\Phi(5/3, 1/3; \eta)\right] \qquad (2.9)$$

The derived solution, as well as integral (2.3) provides a simple interpretation, namely, it corresponds to a dipole located in a uniform stream of gas particles with their critical velocity at infinity. The constant $c_2$ is proportional to the moment of the dipole.

Thus the complete system of Eqs. (2.8) yields a solution which defines the field of velocities generated by the perturbation field resulting from viscous interaction between the dipole and the free sonic stream.

Solution (2.9) may, of course, be obtained directly. The initial system of two Eqs. (2.8) is equivalent to a single second order differential equation of function $f_2(\xi)$ which is of the form (2.10)

$$\xi^2 \frac{d^2 f_2}{d\xi^2} + 2\xi\left(\frac{2}{9}\xi^3 - 1\right)\frac{df_2}{d\xi} + 2\left(\frac{2}{3}\xi^3 + 1\right) f_2 = \omega_1(\xi) = -2\omega(\xi) + \xi\frac{d\omega}{d\xi}$$

Having substituted the independent variable in accordance with (2.4), and substituted $f_2^0(\xi) = \eta^{1/3}\psi(\eta)$ into the corresponding homogeneous Eq. (2.10), we find that function $\psi(\eta)$ satisfies the following confluent hypergeometric equation:

$$\eta \frac{d^2\psi}{d\eta^2} + \left(\frac{2}{3} - \eta\right)\frac{d\psi}{d\eta} - \frac{4}{3}\psi = 0$$

written in its canonical form [8]. The two linearly independent integrals of this equation are known, and constitute the (2.9) representation of function $f_2^0(\xi)$.

The knowledge of the linearly independent integrals of the homogeneous equation corresponding to (2.10) makes it possible not only to confirm the validity of Formulas (2.9), but also to find a general solution for the initial system of Eqs. (2.8). The easiest way of achieving this is to resort to the standard method of variation of constants.

We note, first of all, that each of the linear combinations of hypergeometric functions appearing in square brackets in Expressions (2.3) and (2.9) is proportional to the so-called $\Psi$-function [8] originally introduced into mathematical physics by Tricomi.

The asymptotic representation of the $\Psi$-function does not contain exponential terms, when $\eta \to \pm\infty$.

Consequently, we shall select the fundamental system of solutions of the corresponding homogeneous Eq. (2.10) the following integrals:

$$f_2^{01} = \eta^{1/3}\Psi\left(4/3,\ 2/3;\ \eta\right), \qquad f_2^{02} = \eta^{1/3}e^{\eta}\Psi\left(-2/3,\ 2/3;\ -\eta\right)$$

The Wronskian of the selected fundamental system of solutions is of the form

$$W_1 = \frac{8}{9}\exp\left[-\int\left(\frac{4}{9}\xi^2 - \frac{2}{\xi}\right)d\xi\right] = \frac{8}{9}\xi^2 e^{\eta}$$

We now have

$$f_2 = f_2^0(\xi) - f_2^{01}(\xi)\int_{-\infty}^{\xi}\frac{\omega_1(\xi)f_2^{02}(\xi)}{\xi^2 W_1(\xi)}d\xi + f_2^{02}\int_{-\infty}^{\xi}\frac{\omega_1(\xi)f_2^{01}(\xi)}{\xi^2 W_1(\xi)}d\xi \qquad (2.11)$$

With the known expression of function $f_2(\xi)$ we are at once able to write down the formula defining $g_2(\xi)$. Such a formula would, however, be somewhat cumbersome. We may present function $g_2(\xi)$ in a more meaningful form by passing from the initial system (2.8) to the second order differential equation which defines the latter as follows:

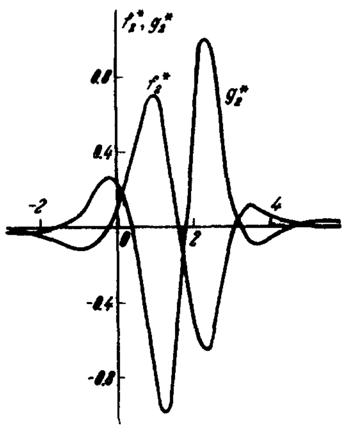

Fig. 1

$$\xi\frac{d^2 g_2}{d\xi^2} + \left(\frac{4}{9}\xi^3 - 1\right)\frac{dg_2}{d\xi} + \frac{20}{9}\xi^2 g_2 = $$
$$= \omega_2(\xi) = -2\omega(\xi) - \frac{2}{3}\xi\frac{d\omega}{d\xi} \qquad (2.12)$$

The fundamental system of solutions of the corresponding homogeneous Eq. (2.12) is written down as follows

$$g_2^{01} = \Psi\left(5/3,\ 1/3;\ \eta\right)$$
$$g_2^{02} = e^{\eta}\Psi\left(-4/3,\ 1/3;\ -\eta\right)$$

with variable $\eta$ defined, as previously by equality (2.4) and by Wronskian

$$W_2 = -\frac{4}{3}2^{1/3}\exp\left[-\int\left(\frac{4}{9}\xi^2 - \frac{1}{\xi}\right)d\xi\right] =$$
$$= -\frac{4}{3}2^{1/3}\xi e^{\eta}$$

We finally obtain

$$g_2 = \left\{1 - \frac{4^{1/3}\Gamma^2(1/3)}{3\Gamma^2(2/3)}\frac{c_1}{c_2}\left[1 + \frac{1}{8}\frac{\Gamma(1/3)}{2^{1/3}\Gamma(2/3)}c_1\right]\right\}g_2^0(\xi) -$$
$$-g_2^{01}(\xi)\int_{-\infty}^{\xi}\frac{\omega_2(\xi)g_2^{02}(\xi)}{\xi W_2(\xi)}d\xi + g_2^{02}(\xi)\int_{-\infty}^{\xi}\frac{\omega_2(\xi)g_2^{01}(\xi)}{\xi W_2(\xi)}d\xi \qquad (2.13)$$

Properties of functions $f_1(\xi)$, $g_1(\xi)$, $f_2^0(\xi)$ and $g_2^0(\xi)$ were analyzed in [1].

Functions $f_2^*(\xi)$ and $g_2^*(\xi)$ which are particular solutions of the nonhomogeneous Eqs.(2.10) and (2.12), respectively, are shown on Fig. 1. A direct integration of the initial system of differential Eqs.(2.8) with constant

$$c_1 = 2^{4/3}\,\Gamma\,(^2/_3)\,[3\Gamma\,(^1/_3)]^{-1}$$

was resorted to for the plotting of these functions.

Conversion with the use of explicit Expressions (2.11) and (2.13) presents considerable difficulties.

The asymptotic expansion of functions here considered defines the behavior of components of the gas particles velocity vector in the neighborhood of the axis of symmetry $r = 0$ and $x < 0$ when $\xi \to -\infty$.

The perturbations induced in a uniform sonic stream by the source and the dipole do not, as previously shown, have singularities all along axis $r = 0$, except at point $x = 0$.

In order to prove this we shall use Formulas (2.3) and (2.9) together with the asymptotic representation of the $\Psi$-function [8]. We have

$$f_1 = \frac{3\Gamma(^1/_3)}{2^{4/3}\Gamma(^2/_3)} c_1 \frac{1}{\xi^2} + \cdots, \quad g_1 = -\frac{9\Gamma(^1/_3)}{2^{4/3}\Gamma(^2/_3)} c_1 \frac{1}{\xi^4} + \cdots$$

$$f_2^0 = -\frac{9\Gamma(^2/_3)}{2\Gamma(^1/_3)} c_2 \frac{1}{\xi^3} + \cdots, \quad g_2^0 = \frac{27\Gamma(^2/_3)}{\Gamma(^1/_3)} c_2 \frac{1}{\xi^5} + \cdots \tag{2.14}$$

The terms of the integral in equality (2.13) tend to vanish when $\xi \to -\infty$ as

$$-\frac{945}{2^{4/3}} \frac{\Gamma(^1/_3)}{\Gamma(^2/_3)} c_1 \left(1 + \frac{1}{20} \frac{\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1\right) \frac{1}{\xi^8} + \cdots$$

therefore the particular solutions $f_2^*(\xi)$ and $g_2^*(\xi)$ of nonhomogeneous Esq.(2.10) and (2.12) are

$$f_2^* = \frac{45\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1 \left[1 + \frac{1}{8} \frac{\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1\right] \frac{1}{\xi^6} + \cdots$$

$$g_2^* = -\frac{18\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1 \left[1 + \frac{1}{8} \frac{\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1\right] \frac{1}{\xi^5} + \cdots$$

It follows from this that perturbations of the longitudinal component of the velocity vector which are due to interaction of the dipole and the source fade with $x \to -\infty$, and $r = 0$ more rapidly than the perturbations in an intrinsic dipole field. As regards the velocity transverse component, the viscous interaction of the two singularities generates perturbations of the same order as those induced by the dipole.

We shall derive the asymptotic expansion of all functions when $\xi \to +\infty$. It appears that the mode of the tendency to zero of functions $f_1(\xi)$, $g_1(\xi)$, $f_2^0(\xi)$ and $g_2^0(\xi)$ is the same as that of relationships (2.14), namely

$$f_1 = -\frac{3\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1 \frac{1}{\xi^2} + \cdots, \quad g_1 = \frac{9\Gamma(^1/_3)}{2^{1/3}\Gamma(^2/_3)} c_1 \frac{1}{\xi^4} + \cdots$$

$$f_2^0 = \frac{9\Gamma(^2/_3)}{\Gamma(^1/_3)} c_2 \frac{1}{\xi^3} + \cdots, \quad g_2^0 = -\frac{54\Gamma(^2/_3)}{\Gamma(^1/_3)} c_2 \frac{1}{\xi^5} + \cdots$$

The behavior of particular solutions $f_2^*(\xi)$ and $g_2^*(\xi)$ when $\xi \to +\infty$ is as follows:

$$f_2^* = \frac{9}{4} \frac{1}{\xi^3} \ln \xi \int_{-\infty}^{+\infty} f_1^2(\xi)\,d\xi + \cdots, \quad g_2^* = \frac{3}{4} \frac{1}{\xi^2} \int_{-\infty}^{+\infty} f_1^2(\xi)\,d\xi + \cdots \tag{2.15}$$

Therefore, in the neighborhood of the axis of symmetry $r \to 0$ and $x > 0$, the velocity components have singularities $v_x \sim x^{-3}$ ln $r$ and $v_r \sim x^{-2}r^{-1}$. The appearance of these is due to the vortex trail, always present in the downstream flow past a body. But the velocity field pattern in vortex trail zone is essentially due to the interaction of the tangential, and not of the normal components of the viscous stress tensor [6]. On the other hand, when the approximate relationship (1.18) is derived from the Navier-Stokes initial equations, then the terms associated with normal stresses and the longitudinal component of the heat flux vector are the predominant factors. Therefore, the relationships of (1.18) and Eqs. (2.1) and (2.6) of the first and second order approximations derived from these are not valid in the narrow zone downsream of the body. Furthermore, if the integrals of solutions of higher orders of approximation did not contain any singularities whatsoever, it would mean that it is in fact possible to derive a solution of the Navier-Stokes system of equations for a flow past a body of finite dimensions without the formation of a vortex trail.

We note that Formulas (2.15) relate to perturbations generated in a uniform stream by a slender body of revolution of cross section $\sigma(x)$ differing from the constant value by const $x^{-1}$, when $x \to +\infty$. In other words, in the theory under consideration the "displacement" in the trail downstream of the body is approximated by

$$\sigma = b_1 + \frac{b_2}{x} + \cdots$$

which conforms with the results of [1]. Constants $b_1$ and $b_2$ in this equality may be expressed in terms of the previously introduced constant $c_1$, while their values are independent of the value of constant $c_2$.

## BIBLIOGRAPHY


1. Ryzhov, O. S., Asymptotic pattern of flow past bodies of revolution in a sonic stream of viscous and heat-conducting gas. PMM Vol. 29, No. 6, 1965.
2. Guderley, K. G. and Yoshihara, H., An axial-symmetric transonic flow pattern. Quart. Appl. Math., Vol. 8, No. 4, 1951
3. Barish, D. T. and Guderley, K. G., Asymptotic forms of shock waves in flows over symmetrical bodies at Mach 1. J. Aeronaut. Sci., Vol. 20, No. 7, 1953.
4. Falkovich, S. V. and Chernov, I. A., Flow of a sonic gas stream past a body of revolution. PMM Vol. 28, No. 2, 1964.
5. Müller, E. A. and Matschat, K., Ähnlichkeitslösungen der transsonischen Gleichungen bei der Anström-Machzahl 1. Proc. 11th Internat. Congr. of Appl. Mech., Munich, 1964.
6. Landau, L. D. and Lifshits, E. M., Mechanics of Continuous Media. 2nd edition, Gostekhizdat, 1954.
7. Landau, L. D. and Lifshits, E. M., Statistical Physics. M., Izd. "Nauka", 1964.
8. Higher Transcendental Functions. Vol. 1, McGraw-Hill, N.Y., Toronto-L. 1953.


Translated by J. J. D.